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A Singular Riesz Product in the Nevai Class and Inner Functions with the Schur Parameters in $\bigcap_{p>2} l^p$

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TO THE MEMORY OF TEACHER, FRIEND, AND CO-AUTHOR,
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There exist singular Riesz products $d\sigma = \prod_{\kappa=1}^{\infty} (1 + \mathrm{Re}(\alpha_\kappa \zeta^{n_\kappa}))$ on the unit circle $\mathbb{T}$ with the parameters $(a_n)_{n \geqslant 0}$ of orthogonal polynomials in $L^2(d\sigma)$ satisfying $\sum_{n=0}^{\infty} |a_n|^p < +\infty$ for every p, $p > 2$. The Schur parameters of the inner factor of the Cauchy integral $\int_{\mathbb{T}} (\zeta - z)^{-1} \, d\sigma(\zeta)$, σ being such a Riesz product, belong to $\bigcap_{p>2} l^p$. © 2001 Academic Press

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1. Given a probability measure σ on $\mathbb{T} = \{\zeta \colon |\zeta| = 1\}$ the orthogonal polynomials $(\varphi_n)_{n \geqslant 0}$ in $L^2(d\sigma)$ are defined by

$$\begin{aligned} &\varphi_n(z) = k_n \cdot z^n + \cdots + \varphi_n(0), \qquad k_n > 0, \\ &\int_{\mathbb{T}} \varphi_i \bar{\varphi}_j \, d\sigma = \delta_{ij}. \end{aligned} \tag{1.1}$$

For a polynomial p of degree n in z we put $p^*(z) = z^n \overline{p(1/\bar{z})}$. It follows from the recurrence formulae (see [7])

$$\begin{aligned} k_n \varphi_{n+1} &= k_{n+1} z \varphi_n + \varphi_{n+1}(0) \, \varphi_n^* \\ k_n \varphi_{n+1}^* &= k_{n+1} \varphi_n^* + \overline{\varphi_{n+1}(0)} \, z \varphi_n \end{aligned} \tag{1.2}$$

that the orthogonal polynomials are uniquely determined by their parameters $a_n = -\overline{\varphi_{n+1}(0)}/k_{n+1}$, $n = 0, 1, \ldots$. We call $(a_n)_{n \geqslant 0}$ the *Geronimus parameters* of σ.

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The Herglotz formula

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\sigma(\zeta) = \frac{1+zf}{1-zf}, \qquad |z|<1, \tag{1.3}$$

establishes a one-to-one correspondence between probability measures on $\mathbb{T}$ and contractive holomorphic functions in the unit disc $\mathbb{D} = \{z: |z|<1\}$, i.e., the points of the unit ball $\mathscr{B}$ of the Hardy algebra H^∞.

For f, $f \in \mathscr{B}$, the Schur algorithm is defined by

$$f(z) \stackrel{\text{def}}{=} f_0(z) = \frac{zf_1(z)+\gamma_0}{1+\bar{\gamma}_0 zf_1(z)}; \dots f_n(z) = \frac{zf_{n+1}(z)+\gamma_n}{1+\bar{\gamma}_n zf_{n+1}(z)}; \dots . \tag{1.4}$$

If f is not a finite Blaschke product then the sequence $\gamma_n = f_n(0)$, $n=0, 1, \dots$, of *the Schur parameters* of f is infinite.

THEOREM (Geronimus [3, 4]). *The Geronimus parameters of a probability measure σ on $\mathbb{T}$ coincide with the Schur parameters of the function f related with σ by* (1.3).

It is clear from (1.3) (take the real parts) and from Fatou's theorem on nontangential limits that f is an inner function if and only if σ is a singular measure. In addition $(1-zf)^{-1}$ is an outer function for every f in $\mathscr{B}$. Subtracting 1 from the both sides of (1.3), we obtain that

$$\int_{\mathbb{T}} \frac{d\sigma(\zeta)}{\zeta-z} = f(z)\cdot(1-zf)^{-1}, \qquad |z|<1, \tag{1.5}$$

is the canonical Nevanlinna factorization of the Cauchy integral of σ into the product of the inner function f and the outer function $(1-zf)^{-1}$.

2. The first example of a singular measure σ with $\lim_n a_n = 0$ was constructed by D. Lubinsky [6]. In the present paper we show that the Nevai class contains also singular Riesz products which are very close to measures satisfying the Szegő condition

$$\begin{aligned}\prod_{n=0}^{\infty} (1-|a_n|^2) &= \exp\left\{\int_{\mathbb{T}} \log(\sigma')\, dm\right\} \\ &= \exp\left\{\int_{\mathbb{T}} \log(1-|f|^2)\, dm\right\} > 0\end{aligned} \tag{2.1}$$

(see [2]). Notice that, by (2.1) f cannot be an inner function if $\sum_{n=0}^{\infty} |\gamma_n|^2 < +\infty$.

We recall (see [8]) that a Riesz product is a probability measure σ on $\mathbb{T}$ with the formal Fourier series defined by the infinite product

$$d\sigma \sim \prod_{\kappa=1}^{\infty} (1 + \operatorname{Re}(\alpha_\kappa \zeta^{n_\kappa})), \tag{2.2}$$

where $0 < |\alpha_\kappa| \leqslant 1$, $\kappa = 1, 2, \ldots$.

In what follows we assume that

$$n_{\kappa+1} > 2(n_\kappa + n_{\kappa-1} + \cdots + n_1), \tag{2.3}$$

$$\sum_{\kappa=1}^{\infty} |\alpha_\kappa|^{2p} < +\infty \qquad \text{for every} \quad p,\ p > 1, \tag{2.4}$$

$$\sum_{\kappa=1}^{\infty} |\alpha_k|^2 = +\infty. \tag{2.5}$$

Condition (2.3) says that every non-zero Fourier coefficient $\hat{\sigma}(\kappa)$, $\kappa \neq 0$, is a product of a finite number of multipliers $\alpha_j/2$ with different indices j. This together with (2.4) implies that $(\hat{\sigma}(\kappa))_{\kappa \in \mathbb{Z}} \in \bigcap_{p>2} l^p$. Condition (2.5) implies that σ is a singular measure (see [8]).

THEOREM 1. *There exists a singular Riesz product σ with the Geronimus parameters $(a_n)_{n \geqslant 0}$ satisfying*

$$\sum_{n=0}^{\infty} |a_n|^p < +\infty \tag{2.6}$$

for every p, $p > 2$.

Proof. We construct the required measure in the class of Riesz products σ, satisfying (2.3)–(2.5), by specifying the growth of n_κ.

Suppose that the numbers $n_1, \ldots, n_\kappa$ are already choosen. The partial product

$$p_\kappa(\zeta) = \prod_{j=1}^{\kappa} (1 + \operatorname{Re}(\alpha_j \zeta^{n_j})), \qquad \zeta \in \mathbb{T} \tag{2.7}$$

is non-negative on $\mathbb{T}$. Therefore by the Feijer theorem [7] we have $p_\kappa = |h_\kappa|^2$, where h_κ is a polynomial of degree $n_1 + \cdots + n_\kappa$, which has no zeros in $\mathbb{D}$. Hence the Fourier spectrum $\operatorname{spec}(\sigma_\kappa)$ of the probability measure $d\sigma_\kappa = |h_\kappa|^2\, dm$ lies in the segment $[-\mathcal{N}, \mathcal{N}]$, $\mathcal{N} = n_1 + \cdots + n_\kappa$, and σ_κ is a Szegő measure (see (2.1)).

Not specifying the choice of $n_{\kappa+1}$ yet, we notice that by (2.3) and (2.7) the measure $d\sigma_{\kappa+1}$ is a linear combination of three measures with disjoint Fourier spectra

$$d\sigma_{\kappa+1} = (\bar{\alpha}_{\kappa+1}/2)\,\bar{\zeta}^{n_{\kappa+1}}\,d\sigma_\kappa + d\sigma_\kappa + (\alpha_{\kappa+1}/2)\,\zeta^{n_{\kappa+1}}\,d\sigma_\kappa.$$

Indeed, $\operatorname{spec}(\zeta^{n_{\kappa+1}}\,d\sigma_\kappa) \subset [n_{\kappa+1} - \mathcal{N}, n_{\kappa+1} + \mathcal{N}]$ and $\operatorname{spec}(\bar{\zeta}^{n_{\kappa+1}}\,d\sigma_\kappa) \subset [-n_{\kappa+1} - \mathcal{N}, -n_{\kappa+1} + \mathcal{N}]$. From this we conclude that the Hilbert spaces $L^2(d\sigma_\kappa)$ and $L^2(d\sigma_{\kappa+1})$ (and, consequently, $L^2(d\sigma)$) induce identical inner products on the subspace $\mathcal{P}_n$ of polynomials in z of degree n for $n < n_{\kappa+1} - \mathcal{N}$.

Recall that the orthogonal polynomials are obtained by the application of the Gram–Schmidt orthogonalization algorithm to the family of monomials $(z^\kappa)_{\kappa \geqslant 0}$. Therefore the polynomials $\varphi_0, \varphi_1, ..., \varphi_n$ in $\mathcal{P}_n$ are orthogonal in $L^2(d\sigma_\kappa)$, $L^2(d\sigma_{\kappa+1})$, and in $L^2(d\sigma)$ for $n < n_{\kappa+1} - \mathcal{N}$. It follows that our future choice of $n_{\kappa+1}$ cannot influence the values of the Geronimus parameters a_n of σ with $n \leqslant n_1 + \cdots + n_\kappa = \mathcal{N} < n_{\kappa+1} - \mathcal{N}$.

Keep for a moment the notation a_n for the parameters of σ_κ. We can apply (2.1) to the Szegő measure σ_κ and find an integer $\mathcal{N}_\kappa$, satisfying $\mathcal{N}_\kappa > n_1 + \cdots + n_\kappa$ and

$$\begin{aligned} &\exp\left\{\int_{\mathbb{T}} \log(\sigma_\kappa')\,dm\right\} \\ &\quad \leqslant \prod_{j=0}^{\mathcal{N}_\kappa} (1 - |a_j|^2) \leqslant (1 + |\alpha_{\kappa+1}^2|) \exp\left\{\int_{\mathbb{T}} \log(\sigma_\kappa')\,dm\right\}. \end{aligned} \tag{2.8}$$

We put $n_{\kappa+1} = 2\mathcal{N}_\kappa$. Since $n_{\kappa+1} - \mathcal{N} > 2\mathcal{N}_\kappa - \mathcal{N}_\kappa = \mathcal{N}_\kappa$, we obtain that the polynomials $\varphi_0, ..., \varphi_{\mathcal{N}_\kappa}$ are orthogonal both in $L^2(d\sigma_\kappa)$ and in $L^2(d\sigma)$.

To prove that the measure σ obtained satisfied (2.6) we need the following elementary lemma.

Lemma 1. *Let* $0 \leqslant \alpha \leqslant 1$ *and* n *be an arbitrary integer. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1 + \alpha \cos nx)\,dx = \log\frac{1}{1 + a^2}, \tag{2.9}$$

where $a = \alpha(1 + \sqrt{1 - \alpha^2})^{-1}$.

Proof. We consider the polynomial $p(z) = (1 + az^n)/\sqrt{1 + a^2}$, which does not vanish in $\mathbb{D}$, and use the mean-value property of the harmonic function $\log|P(z)|^2$. ∎

By (2.8) we have

$$\prod_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} (1-|a_j|^2)$$

$$= \prod_{j=0}^{\mathcal{N}_{\kappa+1}} (1-|a_j|^2) \cdot \prod_{j=0}^{\mathcal{N}_\kappa} (1-|a_j|^2)^{-1}$$

$$\geqslant \frac{1}{1+|\alpha_{\kappa+1}|^2} \cdot \exp\left\{\int_{\mathbb{T}} \log\left(\frac{\sigma'_{\kappa+1}}{\sigma'_\kappa}\right) dm\right\}$$

$$= \frac{1}{1+|\alpha_{\kappa+1}|^2} \cdot \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1+|\alpha_{\kappa+1}| \cos(n_{\kappa+1}\theta+\varphi_{\kappa+1}))\, d\theta\right\},$$

where $\varphi_{\kappa+1} = \arg \alpha_{\kappa+1}$. By Lemma 1 this implies that

$$\prod_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} (1-|a_j|^2) \geqslant \frac{1}{(1+|\alpha^2_{\kappa+1}|)^2}. \tag{2.10}$$

It follows that

$$\sum_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} |a_j|^2 \leqslant -\sum_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} \log(1-|a_j|^2) \leqslant 2\log(1+|\alpha^2_{\kappa+1}|) \leqslant 2\,|\alpha^2_{\kappa+1}|$$

and finally for every p, $p>1$,

$$\sum_{j=\mathcal{N}_1+1}^{\infty} |a_j|^{2p} = \sum_{\kappa=1}^{\infty} \left(\sum_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} |a_j|^{2p}\right)$$

$$\leqslant \sum_{\kappa=1}^{\infty} \left(\sum_{j=\mathcal{N}_\kappa+1}^{\mathcal{N}_{\kappa+1}} |a_j|^2\right)^p \leqslant 2^p \sum_{\kappa=1}^{\infty} |\alpha_{\kappa+1}|^{2p},$$

which obviously yields (2.6). ∎

It is easy to estimate the growth of $\|\varphi_n\|_\infty = \sup_{\mathbb{T}} |\varphi_n|$ for the singular measures σ obtained. It follows from (1.2) that

$$\left\|\frac{\varphi^*_{n+1}}{\varphi^*_n} - 1\right\|_\infty = |a_n|\,(1+o(1)).$$

Therefore

$$\|\varphi_{n+1}\|_\infty \leqslant \prod_{\kappa=0}^{n} \left\|\frac{\varphi^*_{\kappa+1}}{\varphi^*_\kappa}\right\|_\infty \leqslant \prod_{\kappa=0}^{n} (1+\mathscr{C}\cdot|a_\kappa|)$$

$$\leqslant \exp\left\{\mathscr{C}\cdot\sum_{\kappa=0}^{n} |a_\kappa|\right\} \leqslant e^{\mathscr{C}_q n^{1/q}}$$

for every q, $q<2$, by the Hölder inequality.

It follows from (2.6) by the Geronimus theorem that the Schur parameters $(\gamma_n)_{n\geqslant 0}$ of the inner function f in (1.5) satisfy

$$\sum_{n=0}^{\infty} |\gamma_n|^p < +\infty \tag{2.11}$$

for every p, $p > 2$.

COROLLARY 1. *There exists a Blaschke product with the Schur parameters satisfying* (2.11) *for every* p, $p > 2$.

Proof. For α, $|\alpha| < 1$, we define the Möbius transform $\tau_\alpha(z)$ of $\mathbb{D}$ by $\tau_\alpha(z) = (z-\alpha)\cdot(1-\bar{\alpha}z)^{-1}$. By the Frostman theorem [1] for all α, $|\alpha| < 1$, except possibly for a set of logarithmic capacity zero the function $f_\alpha = \tau_\alpha \circ f$ is a Blaschke product. Put $\lambda_\alpha = (1-\alpha\bar{\gamma}_0)(1-\bar{\alpha}\gamma_0)^{-1}$. Clearly $|\lambda_\alpha| = 1$ and easy algebra shows that

$$f_\alpha(z) = \frac{\theta\tau_\alpha(\gamma_0) + z\lambda_\alpha f_1(z)}{1 + \overline{\theta\tau_\alpha(\gamma_0)}\cdot z\lambda_\alpha f_1(z)}.$$

Therefore the Schur parameters of f_α are $\theta\tau_\alpha(\gamma_0)$, $\lambda_\alpha\gamma_1$, $\lambda_\alpha\gamma_2$, ..., $\lambda_\alpha\gamma_n$, ... ∎

In the conclusion we notice that some information on the behavior of the Schur parameters of general inner functions can be captured from the following result of Holland [5].

THEOREM. *Let σ be a singular probability measure on $\mathbb{T}$ and let f be an inner function satisfying* (1.3). *Then*

$$\int_{\mathbb{T}} \left| 1 - \sum_{\kappa=0}^{n-1} \hat{f}(\kappa)\, z^{\kappa+1} \right|^2 d\sigma = \sum_{\kappa=n}^{\infty} |\hat{f}(\kappa)|^2.$$

Since $k_n^{-2} = \inf\{\int_{\mathbb{T}} |p_n|^2\, d\sigma\colon p_n(0) = 1,\ p_n \in \mathcal{P}_n\}$ we obtain

$$\prod_{\kappa=0}^{n-1} (1-|\gamma_\kappa|^2) = k_n^{-2} \leqslant \sum_{\kappa=n}^{\infty} |\hat{f}(\kappa)|^2, \tag{2.12}$$

which again shows that $\sum_{n=0}^{\infty} |\gamma_n|^2 = +\infty$ for an inner function f.

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